Lozenge-tiling Markov Chain: Lattice Paths, Contraction Property and Wilson's Method

MATH GR 6153 Probability II Final Presentation

Wilson, David Bruce. "Mixing times of lozenge tiling and card shuffling Markov chains"

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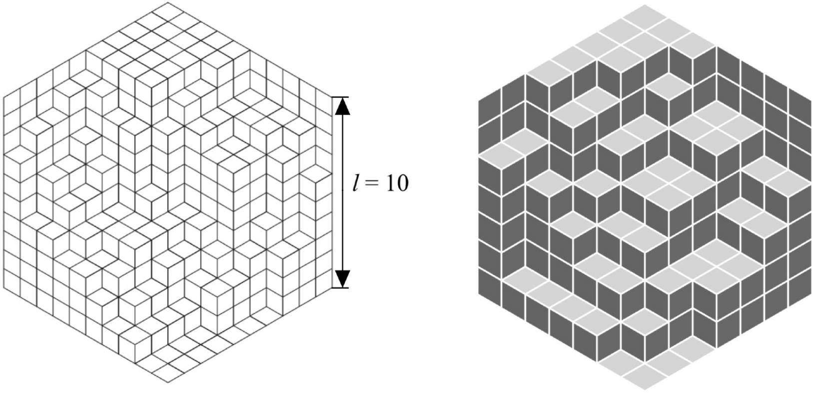
Luby-Randall-Sinclair Markov chain

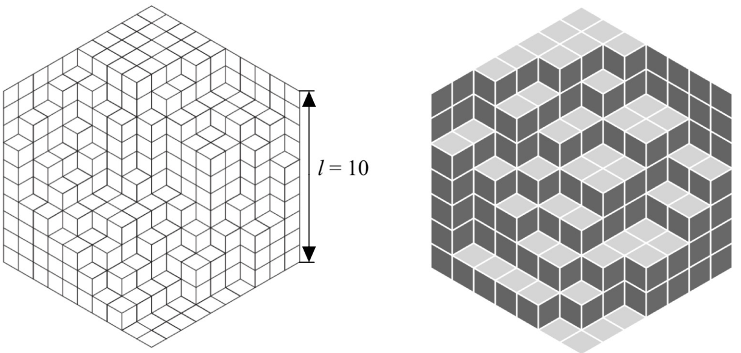
State Space



This Markov chain was introduced by Luby, Randall, and Sinclair, where each state can be represented by lattice paths.
The tiled region is assumed to be simply connected.
The state space is the set of all workable tiling ways, which is equivalent to a group of lattice paths.

Luby-Randall-Sinclair Markov chain

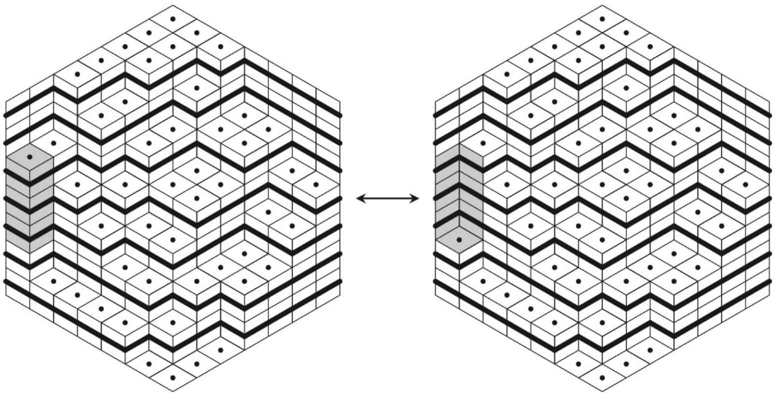
Transition Rule



The transition rule is to uniformly pick a point on a lattice path and refresh it, except when refresh is not allowed because the operation will make the new path overlap the path bounded above (below).

LUBY-RANDALL-SINCLAIR MARKOV CHAIN

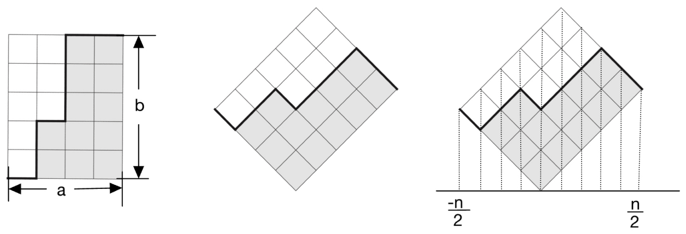
NONLOCAL MOVE



If this occurs, a "nonlocal move" is adopted. If there are k chains bounded above (below) the local minimum (maximum), and we decide to move the point up (down), then with a probability of $\frac{1}{k+1}$, we do so, and with a probability of $\frac{k}{k+1}$, we do nothing.

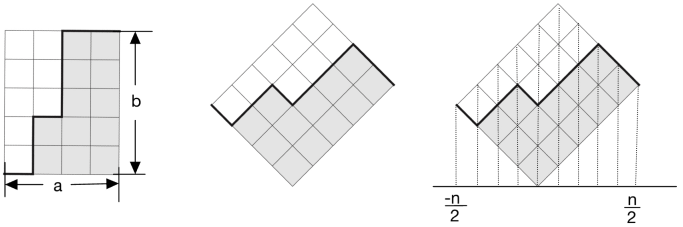
LUBY-RANDALL-SINCLAIR MARKOV CHAIN

HEIGHT FUNCTION



The region where the lattice paths reside, which is equivalent to the tiled region, is assumed to have a width of n, with m local moves separating the top and bottom configurations, and p points where a lattice path may be moved. We define the height function of path i

$$h_i(-\frac{n}{2}) = 0, \ \textit{and} \ h_i(x) = \begin{cases} h_i(x-\frac{1}{2})+\frac{1}{2}, & \textit{an up move from } x-\frac{1}{2} \textit{ to } x, \\ h_i(x-\frac{1}{2})-\frac{1}{2}, & \textit{otherwise} \end{cases} \tag{1}$$



The displacement function of a tiling is:

$$\Phi(h) = \sum_{i} \sum_{x=-\frac{n}{2}}^{\frac{n}{2}} h_i(x) cos \frac{\beta x}{n}, \ \beta \in [0, \pi]. \tag{2}$$

The gap function between two ordered tilings is $\Phi(g) = \Phi(\hat{h}) - \Phi(\check{h})$, where $\hat{h}$ and $\check{h}$ represent the height functions of the tilings, and $g = \hat{h} - \check{h}$.
$\Phi(g) = 0$ if and only if $\hat{h}$ and $\check{h}$ completely overlap.

Given the condition that the location x at path i is picked, we have that

$$\mathbb{E}[\Delta\Phi(g)|x,i] \leq [\frac{g_i(x-\frac{1}{2})+g_i(x+\frac{1}{2})}{2} - g_i(x)]cos(\frac{\beta x}{n}) := B(g,x,i). \tag{3}$$

And with some algebra, we can construct the contraction property in this situation, which is

$$\mathbb{E}[\Delta\Phi] \leq \frac{-1+cos(\frac{\beta}{n})}{p}\Phi, \tag{4}$$

and the equality holds when $\beta = \pi$. We point that

$$-\frac{\beta^2}{2n^2p} \leq \frac{-1+cos\frac{\beta}{n}}{p} \leq -\frac{\beta}{2n^2p}, \tag{5}$$

Using this property, we obtain that

$$t_{mix}(\epsilon) \leq \frac{2 + o(1)}{\beta^2} pn^2 log(\frac{m}{\Phi_{min}\epsilon}). \tag{6}$$

The sketch of the proof is that

$$\mathbb{P}(\Phi_t > 0)\Phi_{min} \leq \mathbb{E}[\Phi_t] \leq \Phi_0[1 - \frac{1 - cos\frac{\beta}{n}}{p}]^t \leq \Phi_0[1 - \frac{\beta^2}{2n^2p}]^t \leq \Phi_0 e^{-\frac{\beta^2}{2n^2p}t} \tag{7}$$

where $\Phi_{min} = cos[\beta\frac{\frac{n}{2}-1}{n}]$ the lowest positive gap. We have $\Phi_0 < m$ and $\Phi_{min} > cos\frac{\beta}{2} \sim \frac{\pi-\beta}{2}$.
Wilson further argued that the optimal way to lower the upper bound is to let $\beta = \pi - \Theta(\frac{1}{logn})$, which leads that

$$t_{mix}(\epsilon) \leq \frac{2 + o(1)}{\pi^2} pn^2 log(\frac{m}{\epsilon}). \tag{8}$$

Upper Bound
Hexagon

When the tiling region is a regular hexagon with side length l, we have that $n = 2l$, $m = l^3$ and $p = 2l(l-1)$. The upper bound for $t_{mix}(\epsilon)$ is

$$\frac{48 + o(1)}{\pi^2} l^4 log(l).$$

Lemma 1

Let (X_t) be an irreducible aperiodic Markov chain with state space χ and transition matrix P. Let Φ be an eigenfunction of P with eigenvalue $\frac{1}{2} < \lambda < 1$, For $0 < \epsilon < 1$ and let $R > 0$ satisfy $\mathbb{E}_x(|\Phi(X_1) - \Phi(x)|^2) \leq R$ for all $x \in \chi$. Then for any x,

$$t_{mix}(\epsilon) \geq \frac{1}{2log(\frac{1}{\lambda})}[log(\frac{(1 - \lambda\Phi^2(x))}{2R} + log(\frac{1 - \epsilon}{\epsilon}))]. \quad (9)$$

Lemma 2

If a function Φ on the state space of a Markov chain satisfies

$$\mathbb{E}(\Phi(X_{t+1})|X_t) = (1-\gamma)\Phi(X_t), \tag{10}$$

and $\mathbb{E}[(\Delta\Phi)^2|X_t] \leq R$, where $\Delta\Phi = \Phi(X_{t+1}) - \Phi(X_t)$, then when the number of move t is bounded by

$$\frac{log(\Phi_{max}) + \frac{1}{2}log(\frac{\gamma\epsilon}{4R})}{-log(1-\gamma)}, \tag{11}$$

and

$$either\, 0 \leq \gamma \leq 2-\sqrt{2},\; or\, 0 < \gamma \leq 1\, with\, odd\, t, \tag{12}$$

we have $d(t) \geq 1-\epsilon$.

Lower Bound
Hexagon

As l becomes large, $\mathbb{E}(\Phi(X_{t+1})|X_t) = (1-\gamma)\Phi(X_t)$ is satisfied, where $\gamma \sim \frac{\beta^2}{2pn^2} \sim \frac{\pi^2}{16l^4}$ and $R \leq l$ due to the nonlocal move. Therefore, according to Wilson's method, the lower bound is

$$\frac{8-o(1)}{\pi^2} l^4 \log(l).$$

Reference

[1] Wilson, David Bruce. “Mixing times of lozenge tiling and card shuffling Markov chains”. In: The Annals of Applied Probability 14.1 (2004), pp. 274–325.

[2] Levin, David A and Peres, Yuval. Markov chains and mixing times. Vol. 107. American Mathematical Soc., 2017.18